

ULAM'S METHOD FOR LASOTA-YORKE MAPS WITH HOLES

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ABSTRACT. Ulam's method is a rigorous numerical scheme for approximating invariant densities of dynamical systems. The phase space is partitioned into connected sets and an inter-set transition matrix is computed from the dynamics; an approximate invariant density is read off as the leading left eigenvector of this matrix. When a hole in phase space is introduced, one instead searches for *conditional* invariant densities and their associated escape rates. For Lasota-Yorke maps with holes we prove that a simple adaptation of the standard Ulam scheme provides convergent sequences of escape rates (from the leading eigenvalue), conditional invariant densities (from the corresponding left eigenvector), and quasi-conformal measures (from the corresponding right eigenvector). We also immediately obtain a convergent sequence for the invariant measure supported on the survivor set. Our approach allows us to consider relatively large holes. We illustrate the approach with several families of examples, including a class of Lorenz maps.

1. INTRODUCTION

Dynamical systems $\hat{T} : I \rightarrow I$ typically model complicated deterministic processes on a phase space I . The map \hat{T} induces a natural action on probability measures η on I via $\eta \mapsto \eta \circ \hat{T}^{-1}$. Of particular interest in ergodic theory are those probability measures that are \hat{T} -invariant; that is, η satisfying $\eta = \eta \circ \hat{T}^{-1}$. If η is ergodic, then such η describe the time-asymptotic distribution of orbits of η -almost-all initial points $x \in I$. In this paper, we consider the situation where a “hole” $H_0 \subsetneq I$ is introduced and any orbits of \hat{T} that fall into H_0 terminate. The hole induces an *open dynamical system* $T : X_0 \rightarrow I$, where $X_0 = I \setminus H_0$. Because trajectories are being lost to the hole, in many cases, there is no T -invariant probability measure. One can, however, consider *conditionally invariant* probability measures [26], which satisfy $\eta \circ T^{-1}(I) \cdot \eta = \eta \circ T^{-1}$, where $0 < \eta \circ T^{-1}(I) < 1$ is identified as the *escape rate* for the open system.

We will study \hat{T} drawn from the class of Lasota-Yorke maps: piecewise C^1 expanding maps of the interval, such that $|D\hat{T}|^{-1}$ has bounded variation. The hole H_0 will be a finite union of intervals. In such a setting, because of the expanding property, one can expect to obtain conditionally invariant probability measures that are *absolutely continuous* with respect to Lebesgue measure [5, 31, 21]. Such conditionally invariant measures are “natural” as they may correspond to the result of repeatedly pushing forward Lebesgue measure by \hat{T} . In the next section we will discuss further conditions due to [21] that make this precise: (i) how much of phase space can “escape” into the hole, and (ii) the growth rate of intervals that partially escape relative to the expansion of the map and the rate of escape. These conditions also guarantee the existence of a *unique* absolutely continuous conditionally invariant probability measure (accim). This accim ν , with density h , and its corresponding escape rate ρ are the first two objects that we will rigorously numerically approximate using Ulam's method. Existence and uniqueness results for subshifts of finite type with Markov holes were previously established by Collet, Martínez and Schmitt in [8]; see also [6, 7, 16].

One may also consider the set of points $X_\infty \subset I$ that never fall into the hole H_0 . A probability measure λ on X_∞ can be defined as the $n \rightarrow \infty$ limit of the accim ν conditioned on X_n . The measure λ will turn out to be the unique \hat{T} -invariant measure supported on X_∞ and has the form $\lambda = h\mu$, where h is a Lebesgue integrable function and μ is known as the *quasi-conformal measure* for \hat{T} . We will also rigorously numerically approximate μ and thus λ . Robustness of these objects with respect to Ulam discretizations is essentially due to a *quasicompactness* property, and a significant part of the paper is devoted to elaborating on this point.

Our main result, Theorem 3.2, concerns convergence properties of an extension of the well-known construction of Ulam [30], which allows for efficient numerical estimation of invariant densities of closed dynamical systems. The Ulam approach partitions the domain I into a collection of connected sets $\{I_1, \dots, I_k\}$ and computes single-step transitions between partition sets, producing the matrix

$$(1) \quad \hat{P}_{ij} = \frac{m(I_i \cap \hat{T}^{-1}I_j)}{m(I_j)}.$$

Li [20] demonstrated that the invariant density of Lasota-Yorke maps can be L^1 -approximated by step functions obtained directly from the leading left eigenvector of \hat{P} . Since the publication of [20] there have been many extensions of Ulam's method to more general classes of maps, including expanding maps in higher dimensions [9, 24], uniformly hyperbolic maps [11, 13], nonuniformly expanding interval maps [25, 14], and random maps [12, 17]. Explicit error bounds have also been developed, eg. [23, 12, 4].

We will show that in order to handle open systems, the definition of \hat{P} above need only be modified to P , having entries

$$(2) \quad P_{ij} = \frac{m(I_i \cap X_0 \cap \hat{T}^{-1}I_j)}{m(I_j)}.$$

As in the closed setting, one uses the leading left eigenvector to produce a step function that approximates the density h of the accim ν . However, in the open setting, the leading eigenvalue of P also approximates the escape rate ρ of ν , and the *right* eigenvector approximates the quasi-conformal measure μ . Note that for closed systems, $\rho = 1$ and $\mu = m$.

The literature concerning the analysis of Ulam's method is now quite large. Early work on Ulam's method for Axiom A repellers [13] showed convergence of an Ulam-type scheme using Markov partitions for the approximation of pressure and equilibrium states with respect to the potential $-\log |\det D\hat{T}|_{E^u}|$. These results apply to the present setting of Lasota-Yorke maps *provided the hole is Markov and projections are done according to a sequence of Markov partitions*. Bahsoun [1] considered non-Markov Lasota-Yorke maps with non-Markov holes and rigorously proved an Ulam-based approximation result for the escape rate. Bahsoun used the perturbative machinery of [19], treating the map T as a small deterministic perturbation of the closed map \hat{T} . In contrast, we apply the perturbative arguments of [19] directly to the open map, considering the Ulam discretization as a small perturbation of T . The advantage of this approach is that we can obtain approximation results whenever the existence results of [21] apply. The latter make assumptions on the expansivity of T (large enough), the escape rate (slow enough), and the rate of generation of "bad" subintervals (small enough). From these assumptions we construct an improved Lasota-Yorke inequality that allows us to get tight enough constants to make applications plausible. Besides estimating the escape rate, we obtain rigorous

L^1 -approximations of the accim and approximations of the quasi-conformal measure that converge weakly to μ . We can treat relatively large holes.

An outline of the paper is as follows. In Section 2 we introduce the Perron-Frobenius operator \mathcal{L} , formally define admissible and Ulam-admissible holes, and develop a strong Lasota-Yorke inequality. Section 3 introduces the new Ulam scheme and states our main Ulam convergence result. Section 4 discusses some specific example maps in detail. Proofs are presented in Section 5.

2. LASOTA-YORKE MAPS WITH HOLES

The following class of interval maps with holes was studied by Liverani and Maume-Deschamps in [21].

Definition 2.1. Let $I = [0, 1]$. We call $\hat{T} : I \curvearrowright$ a Lasota-Yorke map if \hat{T} is a piecewise C^1 map, with finite monotonicity partition ¹ \mathcal{Z} , there exists $\hat{\Theta} < 1$ such that $\|D\hat{T}^{-1}\|_\infty \leq \hat{\Theta}$, and $\hat{g} := |D\hat{T}|^{-1}$ has bounded variation.

The transfer operator for the map \hat{T} is the bounded linear operator $\hat{\mathcal{L}}$, acting on the space BV of functions of bounded variation on I , defined by

$$\hat{\mathcal{L}}f(x) = \sum_{\hat{T}(y)=x} f(y)\hat{g}(y).$$

Definition 2.2. Let $\hat{T} : I \curvearrowright$ be a Lasota-Yorke map. Let $H_0 \subsetneq I$ be a finite union of closed intervals, and let $X_0 = I \setminus H_0$. Let $T : X_0 \rightarrow I$ be the restriction $T = \hat{T}|_{X_0}$. Both T and the pair $T_0 = (\hat{T}, H_0)$ are referred to as open Lasota-Yorke maps (or briefly, open systems), and their associated transfer operator is the bounded linear operator $\mathcal{L} : BV \curvearrowright$ given by

$$\mathcal{L}(f) = \hat{\mathcal{L}}(1_{X_0}f).$$

For each $n \geq 1$, let $X_n = \bigcap_{j=0}^n \hat{T}^{-j}X_0$. Thus, X_n is the set of points that have not escaped by time n . Also, we denote by T^n the function $\hat{T}^n|_{X_{n-1}}$. One can readily check that

$$\mathcal{L}^n(f) = \hat{\mathcal{L}}^n(1_{X_{n-1}}f).$$

Definition 2.3. Let T be an open Lasota-Yorke map. A probability measure ν supported on $X_0 \subset I$ which is absolutely continuous with respect to Lebesgue measure and has density $h = \frac{d\nu}{dm}$ of bounded variation is called an absolutely continuous conditional invariant measure (accim) for T if $\mathcal{L}h = \rho h$ for some $0 < \rho \leq 1$.

A probability measure μ on I which satisfies $\mu(\mathcal{L}f) = \rho\mu(f)$ for every function of bounded variation $f : I \rightarrow \mathbb{R}$, with ρ as above, is called a quasi-conformal measure for T .

Remark 2.4. It is usual to define ν to be an accim if $\nu(A) = \frac{\nu(\hat{T}^{-n}A \cap X_n)}{\nu(X_n)}$ for every $n \geq 0$ and Borel measurable set $A \subset I$. The definitions are indeed equivalent; see [21, Lemma 1.1] for a proof. The same lemma shows that if μ is a quasi-conformal measure for T , then μ is necessarily supported on $X_\infty = \bigcap_{n \geq 0} X_n$. It is also usual to require μ to satisfy $\mu(\mathcal{L}f) = \rho\mu(f)$ for continuous functions only. We will see this makes no difference in our setting, as this weaker requirement implies the stronger one in the previous definition.

¹Throughout this paper, a monotonicity partition \mathcal{Z} refers to a partition such that for every $Z \in \mathcal{Z}$ $\hat{T}|_Z$ has a C^1 extension to \bar{Z} .

2.1. Admissible holes and quasi-invariant measures. As in the work of Liverani and Maume-Deschamps [21], we impose some conditions on the open system in order to be able to analyze it. Let us fix some notation.

Let (\hat{T}, H_0) be an open Lasota-Yorke map, which we also refer to as T . For each $n \geq 1$, let $D_n = \{x \in I : \mathcal{L}^n 1(x) \neq 0\}$, and let $D_\infty := \bigcap_{n \geq 1} D_n$. In what follows, we assume that $D_\infty \neq \emptyset$.

For each $\epsilon > 0$ (not necessarily small), we let $\mathcal{G}_\epsilon = \mathcal{G}_\epsilon(T)$ be the collection of finite partitions of I into intervals such that $\mathcal{Z}_\epsilon \in \mathcal{G}_\epsilon(T)$ if (i) the interior of each $A \in \mathcal{Z}_\epsilon$ is either disjoint from or contained in X_0 , and (ii) for each $A \in \mathcal{Z}_\epsilon$, $\text{var}_A(1_{X_0}|DT^{-1}|) < \|DT^{-1}\|_\infty(1 + \epsilon)$. Since H_0 consists of finitely many intervals, this condition is possible to achieve, as the work of Rychlik [27, Lemma 6] shows. We call \mathcal{G}_ϵ the collection of ϵ -adequate partitions (for T). The set of elements of \mathcal{Z}_ϵ whose interiors are contained in X_0 is denoted by \mathcal{Z}_ϵ^* . Next, the elements of \mathcal{Z}_ϵ^* are divided into *good* and *bad*. A set $A \in \mathcal{Z}_\epsilon^*$ is good if

$$\lim_{n \rightarrow \infty} \inf_{x \in D_n} \frac{\mathcal{L}^n 1_A(x)}{\mathcal{L}^n 1(x)} > 0.$$

We point out that it is shown in [21] that the limit above always exists, as the sequence involved is increasing and bounded, and it is clearly non-negative. The set A is called bad when the limit above is 0. We let

$$\begin{aligned} \mathcal{Z}_{\epsilon,g} &= \{A \in \mathcal{Z}_\epsilon^* : A \text{ is good}\}, \text{ and} \\ \mathcal{Z}_{\epsilon,b} &= \{A \in \mathcal{Z}_\epsilon^* : A \text{ is bad}\}. \end{aligned}$$

Finally, two elements of \mathcal{Z}_ϵ^* are called *contiguous* if there are no other elements of \mathcal{Z}_ϵ^* in between them (but there may be elements of \mathcal{Z}_ϵ that are necessarily contained in H_0). We let $\xi_\epsilon = \xi_\epsilon(T)$ be the infimum over ϵ -adequate partitions for T of the maximum number of contiguous elements in $\mathcal{Z}_{\epsilon,b}$.

In a similar manner, we let $\mathcal{G}_\epsilon^{(n)} = \mathcal{G}_\epsilon^{(n)}(T)$ be the collection of finite partitions of I into intervals such that $\mathcal{Z}_\epsilon^{(n)} \in \mathcal{G}_\epsilon^{(n)}$ if (i) the interior of each $A \in \mathcal{Z}_\epsilon^{(n)}$ is either disjoint from or contained in X_{n-1} , and (ii) for each $A \in \mathcal{Z}_\epsilon^{(n)}$, $\text{var}_A|1_{X_{n-1}}|(DT^n)^{-1}| < \|(DT^n)^{-1}\|_\infty(1 + \epsilon)$. The partitions $\mathcal{Z}_\epsilon^{*(n)}, \mathcal{Z}_{\epsilon,g}^{(n)}, \mathcal{Z}_{\epsilon,b}^{(n)}$ are defined analogously. We denote by $\xi_{\epsilon,n} = \xi_{\epsilon,n}(T)$ the infimum over ϵ -adequate partitions for T^n of the maximum number of contiguous elements in $\mathcal{Z}_{\epsilon,b}^{(n)}$; so $\xi_\epsilon = \xi_{\epsilon,1}$.

The following quantities are relevant in what follows:

$$\begin{aligned} \rho &= \rho(T) := \lim_{n \rightarrow \infty} \inf_{x \in D_n} \frac{\mathcal{L}^{n+1} 1(x)}{\mathcal{L}^n 1(x)}, \\ \tilde{\Theta} &= \tilde{\Theta}(T) := \exp \left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(DT^n)^{-1}\|_\infty \right), \\ \tilde{\xi}_\epsilon &= \tilde{\xi}_\epsilon(T) := \exp \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \log(1 + \xi_{\epsilon,n}) \right), \\ (3) \quad \alpha_\epsilon &= \alpha_\epsilon(T) := \|DT^{-1}\|_\infty(2 + \epsilon + \xi_\epsilon). \end{aligned}$$

Definition 2.5 (Admissible holes). *Let $\hat{T} : I \curvearrowright$ be a Lasota-Yorke map, and $\epsilon > 0$. We say that $H_0 \subset I$ is:*

- an ϵ -admissible hole for \hat{T} if $D_\infty \neq \emptyset$ and $\tilde{\xi}_\epsilon \tilde{\Theta} < \rho$,
- an admissible hole for \hat{T} if it is ϵ -admissible for $\epsilon = 1$ ².

²This is the choice made in [21].

- an ϵ -Ulam-admissible hole for \hat{T} if $D_\infty \neq \emptyset$ and $\alpha_\epsilon < \rho$.

The main result of Liverani and Maume-Deschamps [21] is concerned with the existence of the objects we intend to rigorously numerically approximate.

Theorem 2.6 ([21, Theorem A & Lemma 3.10]). *Assume (\hat{T}, H_0) is an open system with an admissible hole. Then,*

- (1) *There exists a unique absolutely continuous conditionally invariant measure (accim) $\nu = h\mu$ for (\hat{T}, H_0) .*
- (2) *There exists a unique quasi-conformal measure μ for (\hat{T}, H_0) , such that $\mu(\mathcal{L}f) = \rho\mu(f)$ for every $f \in BV$. Furthermore, this measure is atom-free, and satisfies the property that*

$$\mu(f) = \lim_{n \rightarrow \infty} \inf_{x \in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)}$$

for every $f \in BV$, and $\rho = \mu(\mathcal{L}1)$.

- (3) *The measure $\lambda = h\mu$ is, up to scalar multiples, the only T invariant measure supported on X_∞ and absolutely continuous with respect to μ .*
- (4) *There exist $\kappa < 1$ and $C > 0$ such that for any function of bounded variation f ,*

$$\left\| \frac{\mathcal{L}^n f}{\rho^n} - h\mu(f) \right\|_\infty \leq C\kappa^n \|f\|_{BV}.$$

Remark 2.7. *It follows readily from the proof of Theorem 2.6 [21] that the same conclusion can be obtained if the hypothesis of H_0 being an admissible hole is replaced by H_0 being an ϵ -admissible hole for some $\epsilon > 0$.*

To close this section, we present a lemma concerning admissibility of different holes, obtained by enlarging an initial hole H_0 to $H_m := I \setminus X_m$. This broadens the applicability of Theorem 3.2 because enlarging the holes may reduce the number of contiguous bad intervals, and also reduce the variation remaining on the domain of the open Lasota-Yorke map without increasing the expansion.

Lemma 2.8 (Enlarging holes). *Let $T_0 = (\hat{T}, H_0)$ be an open system with an ϵ -admissible hole, and for each $m \geq 0$, let $H_m := I \setminus X_m$. Then, for each $m \geq 0$, $T_m := (\hat{T}, H_m)$ is an open system with an ϵ -admissible hole. Furthermore, let $\rho(T_m), h(T_m)$ and $\mu(T_m)$ be the escape rate, accim and quasi-conformal measures of T_m , respectively. Then we have the following.*

- (1) $\rho(T_m) = \rho(T_0)$,
- (2) $\hat{\mathcal{L}}^m(h(T_m)) = \rho(T_0)^m h(T_0)$, and
- (3) $\mu(T_m) = \mu(T_0)$.

The proof of Lemma 2.8 is presented in §5.3.

3. ULAM'S METHOD FOR LASOTA-YORKE MAPS WITH HOLES

3.1. The Ulam scheme. In the case of a *closed* system \hat{T} , the well-known Ulam method introduced in [30] provides a way of approximating the transfer operator with a sequence of finite-rank operators $\hat{\mathcal{L}}_k$ defined as in e.g. [20], each coming from discretizing the interval I into k bins (which may or may not be of equal length). The only requirements are that each bin is a non-trivial interval, and that the maximum diameter of the partition elements, denoted by τ_k , goes to 0 as k goes to infinity. We call such k -bin partition \mathcal{P}_k . The operator $\hat{\mathcal{L}}_k$ preserves the k -dimensional subspace $\text{span}\{\chi_j : \chi_j = 1_{I_j}, I_j \in \mathcal{P}_k\}$.

The matrix \hat{P}_k defined in the introduction represents the action of $\hat{\mathcal{L}}_k$ on this space, with respect to the ordered basis (χ_1, \dots, χ_k) [20].

In the case of an *open* system (\hat{T}, H_0) , one can still follow Ulam's approach to define a discrete approximation \mathcal{L}_k to the transfer operator \mathcal{L} . For a function $f \in BV$, the operator is defined by $\mathcal{L}_k(f) = \pi_k(\mathcal{L}f) = \pi_k \hat{\mathcal{L}}(1_{X_0}f)$, where π_k is given by the formula

$$\pi_k(f) = \sum_{j=1}^k \frac{1}{m(I_j)} \left(\int \chi_j f \, dm \right) \chi_j.$$

The entries of the Ulam transition matrix P_k representing \mathcal{L}_k in the ordered basis (χ_1, \dots, χ_k) are

$$(P_k)_{ij} = \frac{m(I_i \cap X_0 \cap \hat{T}^{-1}I_j)}{m(I_j)}.$$

(When the partition \mathcal{P}_k is uniform³, the transition matrices \hat{P}_k defined in (1) are stochastic, and P_k are substochastic, the loss of mass being a consequence of the presence of a hole.) Since the entries of P_k are non-negative, an extension of the Perron-Frobenius theorem applies and provides the existence of a non-negative eigenvalue $0 \leq \rho_k \leq 1$ of maximal absolute value for P_k , with associated left and right eigenvectors with non-negative entries; see e.g. [3]. In general, these may or may not be unique. Non-negative left eigenvectors \mathbf{p}_k of P_k induce densities on I according to the formula

$$h_k = \sum_{j=1}^k [\mathbf{p}_k]_j \chi_j,$$

(where we adopt the convention that a vector \mathbf{x} can be written in component form as $\mathbf{x} = ([\mathbf{x}]_1, \dots, [\mathbf{x}]_k)$). Non-negative right eigenvectors ψ_k of P_k induce measures μ_k on I according to the formula

$$\mu_k(E) = \sum_{j=1}^k [\psi_k]_j m(I_j \cap E).$$

We conclude the section with the following.

Lemma 3.1. *Let P_k be the matrix representation of $\mathcal{L}_k = \pi_k \circ \mathcal{L}$ with respect to the basis $\{\chi_j\}$. If $P_k \psi_k = \rho_k \psi_k$ then the measure μ_k corresponding to ψ_k satisfies $\mu_k(\mathcal{L}_k \pi_k \varphi) = \rho_k \mu_k(\varphi)$ for every $\varphi \in L^1(m)$.*

Proof. Let $\varphi \in L^1(m)$ and put $\varphi_k = \pi_k \varphi$. Then,

$$\begin{aligned} \mu_k(\varphi) &= \int \varphi \, d\mu_k = \sum_{j=1}^k \int_{I_j} \varphi \, dm [\psi_k]_j = \sum_{j=1}^k \int_{I_j} \pi_k \varphi \, dm [\psi_k]_j \\ &= \sum_{j,j'=1}^k \int_{I_j} \varphi_k \, dm (P_k)_{jj'} [\psi_k]_{j'} (\rho_k)^{-1} = \sum_{j'=1}^k \int_{I_{j'}} \mathcal{L}_k \varphi_k \, dm [\psi_k]_{j'} (\rho_k)^{-1} \\ &= (\rho_k)^{-1} \int \mathcal{L}_k \varphi_k \, d\mu_k = \rho_k^{-1} \mu_k(\mathcal{L}_k \varphi_k), \end{aligned}$$

where the last equality in the second line follows from the fact that P_k is the matrix representing \mathcal{L}_k in the basis $\{\chi_j\}$, and acts on densities by right multiplication (i.e. if \mathbf{p} is the vector representing the function φ_k , then $\mathbf{p}^T P_k$ is the vector representing $\mathcal{L}_k \varphi_k$). \square

³That is, $m(I_i) = m(I_j), \forall i, j$.

3.2. Statement of the main result. The main result of this paper is the following. Its proof is presented in §5.2.

Theorem 3.2. *Let $\hat{T} : I \curvearrowright$ be a Lasota-Yorke map with an ϵ -Ulam-admissible hole H_0 . Let $h \in BV$ be the unique accim for the open system (\hat{T}, H_0) , and μ the unique quasi-conformal measure for the open system supported on X_∞ , as guaranteed by Theorem 2.6. Let ρ be the associated escape rate. For each $k \in \mathbb{N}$, let ρ_k be the leading eigenvalue of the Ulam matrix P_k . Let h_k be densities induced from non-negative left eigenvectors of P_k corresponding to ρ_k . Let μ_k be measures induced from non-negative right eigenvectors of P_k corresponding to ρ_k . Then,*

- (I) *For k sufficiently large, ρ_k is a simple eigenvalue for P_k . Furthermore, $\lim_{k \rightarrow \infty} \rho_k = \rho$, and there exists $\eta \in (0, 1)$ ⁴ such that $|\rho_k - \rho| \leq O(\tau_k^\eta)$, where $\tau_k := \max_{I_j \in \mathcal{P}_k} m(I_j)$ is the maximum diameter of the elements of \mathcal{P}_k .*
- (II) *$\lim_{k \rightarrow \infty} h_k = h$ in $L^1(m)$.*
- (III) *$\lim_{k \rightarrow \infty} \mu_k = \mu$ in the weak- $*$ topology of measures. Furthermore, for every sufficiently large k , $\text{supp}(\mu) \subseteq \text{supp}(\mu_k)$.*

We will also establish a relation between admissibility and Ulam-admissibility of holes.

Lemma 3.3 (Admissibility and Ulam-admissibility). *If H_0 is an ϵ -admissible hole for \hat{T} , there is some $n \in \mathbb{N}$ such that $H_{n-1} := I \setminus X_{n-1}$ is ϵ -Ulam-admissible for \hat{T}^n .*

The proof of this lemma is presented in §5.4. This result, together with Lemma 2.8, broadens the scope of applicability of Theorem 3.2 by allowing to (i) replace the map by an iterate (Lemma 3.3), or (ii) enlarge the hole in a dynamically consistent way (Lemma 2.8). It also ensures that several examples in the literature can be treated with our method; in particular, all the examples presented in [21].

4. EXAMPLES

To illustrate the efficacy of Ulam's method, beyond the small-hole setting, we present some examples of Ulam-admissible open Lasota-Yorke systems. We start with the case of full-branched maps in §4.1, and treat some more general examples, including β -shifts, in §4.2. We then analyze Lorenz-like maps. They provide transparent evidence of the scope of the results for open systems, as well as closed systems with repellers. They also illustrate how the admissibility hypothesis may be checked in applications.

4.1. Full-branched maps. In the next examples, we will use the following notation. Given a Lasota-Yorke map with holes, (\hat{T}, H_0) with monotonicity partition \mathcal{Z} , we let $\mathcal{Z}_h = \{Z \in \mathcal{Z} : Z \subseteq H_0\}$, $\mathcal{Z}_f = \{Z \in \mathcal{Z} : Z \cap H_0 = \emptyset, T(Z) = I\}$ and $\mathcal{Z}_u = \{Z \in \mathcal{Z} : Z \notin \mathcal{Z}_h \cup \mathcal{Z}_f\}$. Thus, the elements of \mathcal{Z}_f are precisely the ones contained in X_0 that are full branches for T , and those of \mathcal{Z}_u are the remaining ones.

Definition 4.1. *A full-branched map with holes, (\hat{T}, H_0) , is a Lasota-Yorke map with holes, such that $\mathcal{Z}_u = \emptyset$.*

For piecewise linear maps, the situation is rather simple.

Lemma 4.2. *Let $T_0 = (\hat{T}, H_0)$ be a piecewise linear full-branched map with holes. Then, for every $\epsilon > 0$ the following holds: $\xi_\epsilon(T_0) = 0$,*

$$\begin{aligned} \rho(T_0) &= 1 - \text{Leb}(H_0), \quad \text{and} \\ \alpha_\epsilon(T_0) &= \max_{Z \in \mathcal{Z}_f} \text{Leb}(Z)(2 + \epsilon). \end{aligned}$$

⁴In fact, any $\eta < \frac{\log \rho / \beta}{-\log \beta}$ with $\rho > \beta > \alpha_\epsilon$ is valid.

Proof. If T_0 is a piecewise linear full-branched map, then each interval $Z \in \mathcal{Z}_f$ is good. Observing that an interval being good is equivalent to having non-zero μ measure, and using the fact that μ is atom-free, each Z may be split into two good intervals Z_-, Z_+ in such a way that there is at most one discontinuity of $g := \mathbf{1}_{X_0} D\hat{T}^{-1}$ on each Z_-, Z_+ . Thus, $\text{var}_{Z_-}(g), \text{var}_{Z_+}(g) \leq \|DT_0^{-1}\|_\infty$. Therefore $\xi_\epsilon(T_0) = 0$. Also,

$$\mathcal{L}_0(1)(x) = \sum_{y \in Z \in \mathcal{Z}_f, T_0(y)=x} \frac{1}{|DT_0(y)|} = \sum_{Z \in \mathcal{Z}_f} \text{Leb}(Z) = 1 - \text{Leb}(H_0).$$

On the other hand, $\sup_{x \in Z \in \mathcal{Z}_f} \frac{1}{|DT_0(x)|} = \max_{Z \in \mathcal{Z}_f} \text{Leb}(Z)$. \square

In fact, in the piecewise linear, full branched setting, a direct calculation shows that Lebesgue measure is an accim for the open system. For perturbations of these systems, explicit estimates of ρ and α_ϵ are not generally available. However, we have the following bounds.

Lemma 4.3. *Let $T_0 = (\hat{T}, H_0)$ be a full-branched map with holes. Then, for every $\epsilon > 0$, there exists some computable $m \in \mathbb{N}$ such that $\xi_\epsilon(T_m) = 0$, where $T_m := (\hat{T}, H_m)$ is obtained from T_0 by enlarging the hole, as in Lemma 2.8. Furthermore,*

$$\begin{aligned} \rho(T_m) = \rho(T_0) &\geq \inf_{x \in I} \sum_{y \in Z \in \mathcal{Z}_f, T_0(y)=x} \frac{1}{|DT_0(y)|} =: \rho_0 \quad \text{and} \\ \alpha_\epsilon(T_m) &\leq \sup_{x \in Z \in \mathcal{Z}_f} \frac{1}{|DT_0(x)|} (2 + \epsilon) =: \alpha_{\epsilon,0}. \end{aligned}$$

An immediate consequence is the following.

Corollary 4.4. *In the setting of Lemma 4.3, if $\rho_0 > \alpha_{\epsilon,0}$, then H_m is ϵ -Ulam admissible for \hat{T} . In this case, Lemma 2.8 allows one to approximate the escape rate, accim and quasi-conformal measure for T_0 via Theorem 3.2 applied to T_m .*

Proof of Lemma 4.3. First, let us note that for any map with $\mathcal{Z}_f \neq \emptyset$, we have that $D_\infty \neq \emptyset$, as the map has at least one fixed point outside the hole. If m is sufficiently large, each interval $Z \in \mathcal{Z}^{(m)}$ is either (i) contained in H_{m-1} , and thus not in $\mathcal{Z}^{*(m)}$ or (ii) $T_0^m(Z) = I$ and $\text{var}_Z(\hat{g}\mathbf{1}_{X_m}) < \|\hat{g}\mathbf{1}_{X_m}\|_\infty(1 + \epsilon)$. In the latter case, Z is a good interval for T_0 , because $\mu_0(Z) = \rho_0^m \mu_0(\mathcal{L}_0^m \mathbf{1}) \geq \rho_0^m \|DT_0^m\|_\infty^{-1} \mu_0(I) > 0$. Since good intervals for T_0 and for T_m coincide (see beginning of proof of Lemma 2.8), we get that $\xi_\epsilon(T_m) = 0$.

Furthermore,

$$\rho(T_0) = \rho(T_0)\mu_0(1) = \mu_0(\mathcal{L}_0(1)) \geq \inf_{x \in I} \mathcal{L}_0(1)(x) = \inf_{x \in I} \sum_{y \in Z \in \mathcal{Z}_f, T_0(y)=x} \frac{1}{|DT_0(y)|}.$$

The bound on $\alpha_\epsilon(T_m)$ follows directly from the definition. \square

The following is an interesting consequence of Lemmas 4.2 and 4.3.

Corollary 4.5. *Let (\hat{T}, H_0) be a piecewise linear full-branched map with holes. Assume that $\text{Leb}(H_0) < 1 - 2 \max_{Z \in \mathcal{Z}_f} \text{Leb}(Z)$. Then, if $\epsilon > 0$ is sufficiently small, H_0 is ϵ -Ulam-admissible for any full-branched map (\hat{S}, H_0) that is a sufficiently small C^{1+Lip} perturbation of (\hat{T}, H_0) (where the C^{1+Lip} topology is defined, for example, by the norm given by the maximum of the C^{1+Lip} norms of each branch). In particular, Theorem 3.2 applies.*

Proof. The statement for (\hat{T}, H_0) follows from Remark 4.2. For perturbations, the statement follows from Lemma 4.3, by observing that the quantities ρ_0 and $\alpha_{\epsilon,0}$, as well as the variation of $1/|D\hat{T}|$ on each interval depend continuously on \hat{T} , with respect to the C^{1+Lip} topology. \square

Corollary 4.5 can apply to maps with arbitrarily large holes, as the next example shows.

Example 4.6 (Arbitrarily large holes). *Let $\delta > 0$, $H_0 = [\delta, 1 - \delta]$, and*

$$T_\delta(x) = \begin{cases} \frac{2}{\delta}x & \text{if } x < \frac{\delta}{2}, \\ 1 - \frac{2}{\delta}(x - \frac{\delta}{2}) & \text{if } \frac{\delta}{2} \leq x < \delta, \\ \frac{2}{\delta}(x - 1 + \delta) & \text{if } 1 - \delta \leq x < 1 - \frac{\delta}{2}, \\ 1 - \frac{2}{\delta}(x - 1 + \frac{\delta}{2}) & \text{if } 1 - \frac{\delta}{2} \leq x \leq 1. \end{cases}$$

Then, $\text{Leb}(H_0) = 1 - 2\delta < 1 - \delta = 1 - 2 \max_{Z \in \mathcal{Z}_f} \text{Leb}(Z)$ and the hypotheses of Corollary 4.5 are satisfied. Thus, Ulam's method converges for sufficiently small C^{1+Lip} perturbations of T_δ that are full-branched.

Other examples of this type may be found in [1] and [2]. Bahsoun [1] established rigorous computable bounds for the errors in the Ulam method, which allowed him to find rigorous bounds on the escape rate for open Lasota-Yorke maps. Bose and Bahsoun [2] related the escape rate to the Lebesgue measure of the hole. Both results rely on the existence of Lasota-Yorke type inequalities, relating BV and $L^1(m)$ norms. Such inequalities may be obtained by exploiting the full-branched structure of the map.

Example 4.7 (Bahsoun [1]). *Let*

$$\hat{T}(x) = \begin{cases} 2.08x & \text{if } x < \frac{1}{2}, \\ 2 - 2x & \text{if } x \geq \frac{1}{2}. \end{cases}$$

In this case, Corollary 4.5 does not yield direct information about the applicability of the Ulam method to perturbations of this map, because $\text{Leb}(H_0) = \frac{.08}{4.16}$ but $1 - 2 \max_{Z \in \mathcal{Z}_f} \text{Leb}(Z) = 0$. However, the hypotheses of the corollary are easily satisfied for the second power of the map. We note that ρ controls the rate of mass loss, which is slower than $4.08/4.16$, while α_ϵ is related to the relaxation rate on the survivor set. In this case, one iteration is not enough to see this asymptotic rate is less than the mass loss. However, two iterations suffice.

4.2. Nearly piecewise linear maps with enough full branches. When non-full branches are present, the dynamics is typically non-Markovian. Thus, even in the piecewise linear setting there may not be direct ways to find the various objects of interest (escape rates, accims and quasi-conformal measures) exactly. We show that Ulam's method provides rigorous approximations in specific systems. The following example is closely related to [21, 6.2 & 6.3].

Lemma 4.8. *Let $T = (\hat{T}, H_0)$ be a piecewise linear Lasota-Yorke map with holes, and assume $\mathcal{Z}_f \neq \emptyset$. Let c_u be the maximum number of contiguous elements in \mathcal{Z}_u . If $\|DT^{-1}\|_\infty(3+c_u) < \rho$, then H_0 is $(1+\epsilon)$ -Ulam-admissible for \hat{T} , for every $\epsilon > 0$ sufficiently small. Thus, the hypotheses of Theorem 3.2 are satisfied.*

Proof. For any map with $\mathcal{Z}_f \neq \emptyset$, we have that $D_\infty \neq \emptyset$, as the map has at least one fixed point outside the hole. Furthermore, for each $Z \in \mathcal{Z}$, one has that $\text{var}_Z(g) \leq 2\|g\|_\infty$, so \mathcal{Z} is a $(1 + \epsilon)$ -adequate partition for T . Also, it follows from the definition of \mathcal{Z}_g that

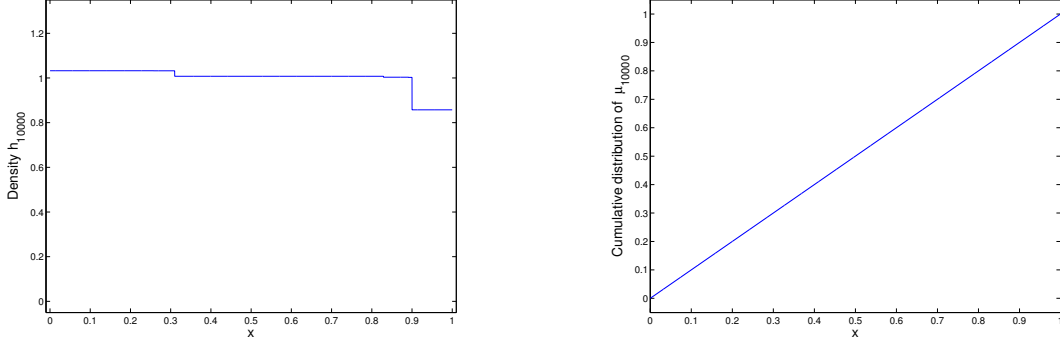


FIGURE 1. Ulam approximations with $k = 10000$ for \hat{T}_β ($\beta = 5.9$) and $H_0 = \emptyset$. Left: Graph of approximate density h_k of acim. Right: The “quasi-conformal” measure, depicted as $\mu_k([0, x])$ vs. x . Note that μ_k approximates Lebesgue measure on $[0, 1]$, as $(\hat{T}_\beta, \emptyset)$ is closed.

$\mathcal{Z}_f \subseteq \mathcal{Z}_g$. Thus, $\mathcal{Z}_b \subseteq \mathcal{Z}_u$, and $\xi_{1+\epsilon} \leq c_u$. Therefore, $\alpha_\epsilon \leq \|DT^{-1}\|_\infty(3 + \epsilon + c_u) < \rho$, provided $\epsilon > 0$ is sufficiently small. \square

A concrete example where the previous lemma applies is that of β -shifts.

Example 4.9. Let $\beta > 1$, and \hat{T}_β be the β -shift, $\hat{T}_\beta(x) = \beta x \pmod{1}$. Let $H_0 \subset I$ be a finite union of closed intervals, and let f be the number of full branches of \hat{T}_β outside H_0 . Then, for the open system (\hat{T}_β, H_0) , we have that $\rho \geq \frac{f}{\beta}$. Then, the hypotheses of Lemma 4.8 are satisfied, provided $f > 3 + c_u$. This happens, for example, when $\beta \geq 5$ and H_0 is a single interval of the form $[\frac{[\beta]}{\beta}, y]$ or $[y, 1]$, with $\frac{[\beta]}{\beta} < y < 1$. Also, when $\beta \geq 6$ and H_0 is a single interval contained in $[\frac{[\beta]}{\beta}, 1]$; or when $\beta \geq 7$ and H_0 is any interval leaving at least 7 full branches in X_0 (recall from Subsection 2.1 that two bad elements of \mathcal{Z}_u are contiguous if there are no good elements of $\mathcal{Z}_f \cup \mathcal{Z}_u$ between them, but there may be elements of \mathcal{Z}_h in between).

We include Figures 1-3, obtained from numerical experiments for $\beta = 5.9$, and two different choices of holes. They include approximations to the densities of accims and cumulative distribution functions of the quasi-conformal measures for systems with a hole, as well as the acim and conformal measure for the closed system.

Remark 4.10. Using lower bounds on ρ such as those of Lemma 4.3, one can extend the conclusion of Lemma 4.8 as in Corollary 4.5, to cover small C^{1+Lip} perturbations of piecewise linear maps that respect the partition $\mathcal{Z}_h \cup \mathcal{Z}_f \cup \mathcal{Z}_u$.

4.3. Lorenz-like maps. Let us consider the following two-parameter family of maps of $I = [-1, 1]$:

$$(4) \quad T_{c,\alpha}(x) = \begin{cases} cx^\alpha - 1 & \text{if } x > 0, \\ 1 - c|x|^\alpha & \text{if } x < 0, \end{cases}$$

where $c > 0, \alpha \in (0, 1)$. When $c > 2$, the system is open and the hole is implicitly defined as $H_{0,c,\alpha} = T_{c,\alpha}^{-1}(\mathbb{R} \setminus [-1, 1])$.

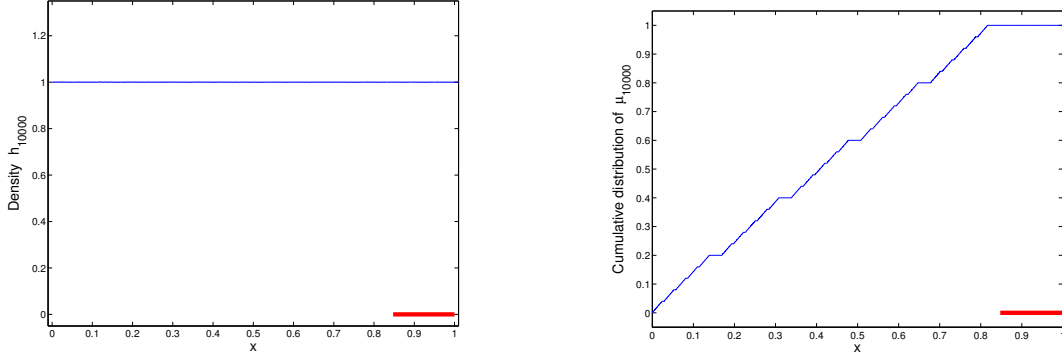


FIGURE 2. Ulam approximations with $k = 10000$ for \hat{T}_β ($\beta = 5.9$) and $H_0 = [\frac{5}{5.9}, 1]$ (shown in red). The computed value of ρ_k is 0.8475 (4 s.f.), and in fact agrees up to 11 s.f. with the exact value for ρ (the length of X_0 : $5/5.9$). Left: Graph of computed density h_k of accim (note that the function 1 is a fixed point of both \mathcal{L} and $\pi_k \mathcal{L}$). Right: The approximate quasi-conformal measure, depicted as $\mu_k([0, x])$ vs. x . Note that μ_k has no support on H_0 .

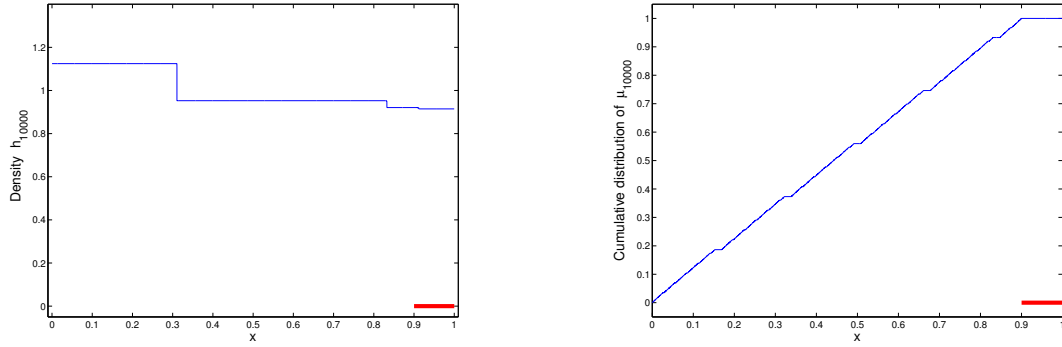


FIGURE 3. Ulam approximations with $k = 10000$ for \hat{T}_β ($\beta = 5.9$) and $H_0 = [0.9001, 1]$ (shown in red). The computed value of ρ_k is 0.9086 (4 s.f.) Left: Graph of approximate density h_k of accim. Right: The approximate quasi-conformal measure, depicted as $\mu_k([0, x])$ vs. x . Note that μ_k has no support on H_0 .

This family of maps has been studied in connection with the famous Lorenz equations,

$$\begin{aligned}
 \dot{x} &= \sigma(y - x) \\
 \dot{y} &= rx - y - xz \\
 \dot{z} &= -bz + xy.
 \end{aligned}
 \tag{5}$$

We take a relatively standard point of view [18, 28, 15], regarding $\sigma = 10$ and $b = 8/3$ as fixed, and r as a parameter. The chaotic attractor discovered by Lorenz [22] at $r = 28$ has since been proved to exist by Tucker [29] (via computer-assisted methods). Its formation is now well understood: A *homoclinic explosion* occurs at $r_{hom} \approx 13.9265$, giving rise to a *chaotic saddle*. As r increases through $r_{het} \approx 24.0579$, heteroclinic connections between $(0, 0, 0)$ and a symmetric pair of periodic orbits Γ^\pm appear and the chaotic saddle becomes an attractor Ω . The orbits Γ^\pm disappear in subcritical Hopf bifurcations

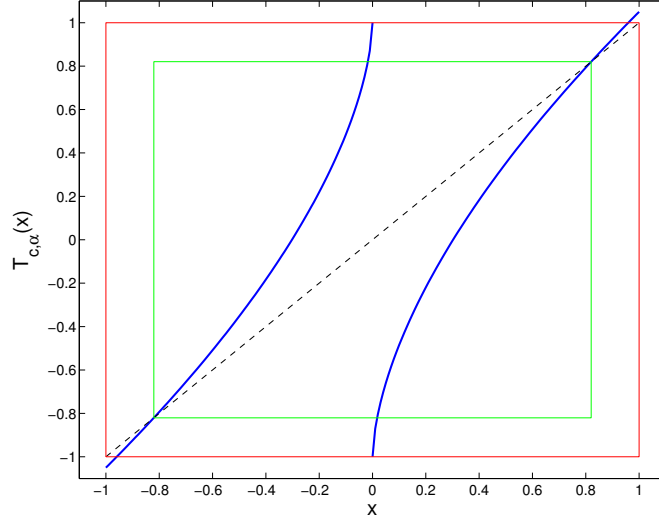


FIGURE 4. Lorenz map $T_{c,\alpha}$, $c = 2.05$, $\alpha = .6$ (blue). Note that $[-1, 1] \subsetneq T_{c,\alpha}[-1, 1]$; bounds of the interval $[-1, 1]$ are depicted in red (the branches of T extend beyond the red box). The chaotic repelling set is confined to the interval between two fixed points (green).

at $r_{Hopf} \approx 24.7368$ (parameter values from [10]). For $r < r_{het}$ almost all orbits are asymptotic to one of two fixed points; for $r_{het} < r < r_{Hopf}$ orbits may approach one of these fixed points, or the attractor Ω ; for $r > r_{Hopf}$ almost all orbits are attracted to Ω .

Maps like (4) model this situation via the following reductions. First, solutions to the ODEs (5) induce a flow on \mathbb{R}^3 ; from this, a *return map* to the section $\Sigma = \{(x, y, z) : z = r - 1\}$ may be constructed. This two-dimensional map is an open dynamical system, since not all orbits return to Σ ⁵. For ‘pre-turbulent’ $r \in (r_{hom}, r_{het})$, the chaotic saddle admits a *strong stable foliation*; the return map to Σ may be further reduced by identifying points on the same stable leaf, resulting in one-dimensional models such as (4). The discontinuity at $x = 0$ corresponds to the intersection of the stable manifold of $(0, 0, 0)$ with Σ ; the exponent $0 < \alpha < 1$ is derived from the eigenvalues of the linearization of the system at the origin, $\alpha = \frac{|\lambda_s|}{\lambda_u}$. The parameter c controls how ‘open’ the map is: when $c \leq 2$, the system is closed, and when $c > 2$, the 1-step survivor set X_0 has the form $X_0 = [-x_{c,\alpha}, x_{c,\alpha}]$, where $x_{c,\alpha} = (2/c)^{1/\alpha}$; this is illustrated in red in Figure 4.

The escape rates of the system $T_{c,\alpha}$ for parameters $0 < \alpha < 1$, $2 < c < 3$ are illustrated in Figure 5. Figure 6 (left) illustrates the cumulative distribution functions of the quasi-conformal measures, $\mu_{c,\alpha}$, for $c = 2.01$ and various values of α . The densities of the accims with respect to Lebesgue are illustrated in Figure 6 for several α values. For $\alpha < 0.5$, the densities become concentrated near the endpoints, as the $\alpha = 0.45$ plot in Figure 6 (right) illustrates.

The escape rate results for these one-dimensional maps can be interpreted coherently with respect to the behaviour of the Lorenz system (5) (although the scenarios differ according to whether $\alpha \leq 1/2$).

- Regarding $T_{c,\alpha}$ as a map on \mathbb{R} , for each value of $\alpha \in (0, 1)$ and $c > 2$ there are two pairs of fixed points: repellers at $\pm y_{c,\alpha} \in (-1, 1)$ (illustrated in green in

⁵For example, the stable manifold to the fixed point $(0, 0, 0)$ intersects Σ , and some orbits of the flow travel directly to $(0, 0, 0)$ after leaving Σ .

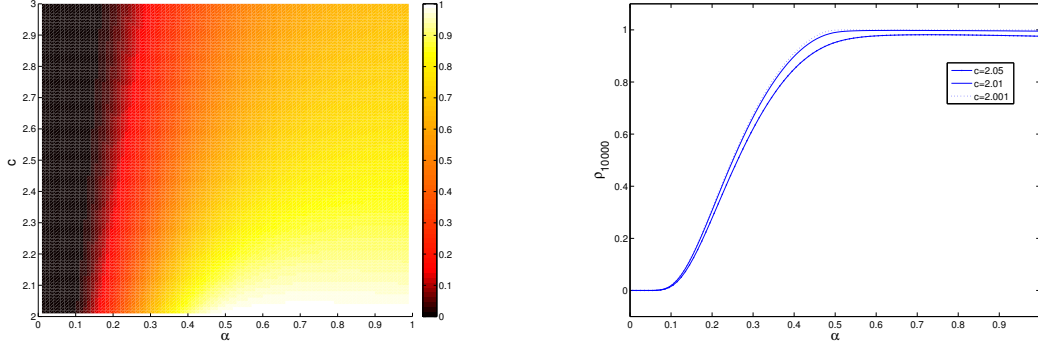


FIGURE 5. Numerical estimate of escape rates via open Ulam method with $k = 10000$ bins. Left: coloured image of leading eigenvalue ρ_{10000} for a range of α and c (light for ρ near 1, dark near 0). Right: ρ_{10000} as a function of α for $c = 2.05, 2.01, 2.001$.

Figure 4) and an attracting *outer* pair $\pm z_{c,\alpha}$ with $|z_{c,\alpha}| > 1$ (beyond the domain of Figure 4). The *inner points* $\pm y_{c,\alpha}$ correspond to the periodic orbits Γ^\pm from the Lorenz flow, and the outer pair correspond to the attracting fixed points of the flow.

- At some $c = c_*(\alpha) \leq 2$ the inner and outer pairs coalesce in a saddle-node bifurcation and for $c < c_*$ the only attractor is a chaotic absolutely continuous invariant measure supported on $[-1, 1]$.
- ($\alpha > 1/2$) Each $T_{c,\alpha}$ is uniformly expanding on X_0 for $c > 2$. For $c > 2$ there is a chaotic repeller in X_0 , and a fully supported accim on $[-1, 1]$. Lebesgue a.e. orbit escapes and is asymptotic to one of the “outer fixed points”. At $c = 2$ the points $\pm x_{c,\alpha} = \pm 1 = \pm y_{c,\alpha}$ become fixed points, with $T'(\pm 1) = 2\alpha > 1$. The open system thus ‘closes up’ as c decreases to 2; this corresponds to the bifurcation point r_{het} in the Lorenz flow (where the origin connects to Γ^\pm). For values of $c < 2$, $T_{c,\alpha}$ admits an acim (which can be accessed numerically by Ulam’s method) and the quasiconformal measure is simply Lebesgue measure. The approach of ρ_k to 1 as $c \rightarrow 2$ can be seen in Figure 5, and the close agreement of $\mu_{2.01,\alpha}$ with Lebesgue measure can be seen in Figure 6 (left) for $\alpha = 0.95$.
- ($\alpha < 1/2$) For $c > 2$, $T_{c,\alpha}$ is open on $[-1, 1]$, but the uniform expansion property fails for c sufficiently close to 2. Indeed, when $c = 2$ the fixed points at ± 1 are the *outer pair* $\pm z_{c,\alpha}$ and $T'(\pm 1) < 1$. For $c \in (c_*, 2)$, these attractors $\pm z_{c,\alpha} \in [-1, 1]$ and coexist with a chaotic repeller in $[-y_{c,\alpha}, y_{c,\alpha}]$. Fortunately, for $c > c_*$ the open system $T_{c,\alpha}$ with hole $I \setminus [-y_{c,\alpha}, y_{c,\alpha}]$ is a Lasota-Yorke map with holes, because it is piecewise expanding⁶. For $c > 2$, our main theorem holds for the application of Ulam’s method to $T_{c,\alpha}$ on $[-y_{c,\alpha}, y_{c,\alpha}]$. However, it is simple to extend this result to $[-1, 1]$: all points in the intervals $\pm(y_{c,\alpha}, 1)$ escape in finitely many iterations, and corresponding cells of the partitions used in Ulam’s method are “transient”. The leading eigenvalue from Ulam’s method and approximate quasiconformal measure on $[-y_{c,\alpha}, y_{c,\alpha}]$ agree with those computed on $[-1, 1]$. The

⁶Corollary 4.4 shows that admissibility of the open system is implied if

$$(6) \quad |T'_{c,\alpha}(y_{c,\alpha})|^{-1} = \sup_{x \in [-y_{c,\alpha}, y_{c,\alpha}]} |T'_{c,\alpha}(x)|^{-1} < \inf \mathcal{L}_{c,\alpha} 1(x).$$

This condition can be verified directly.

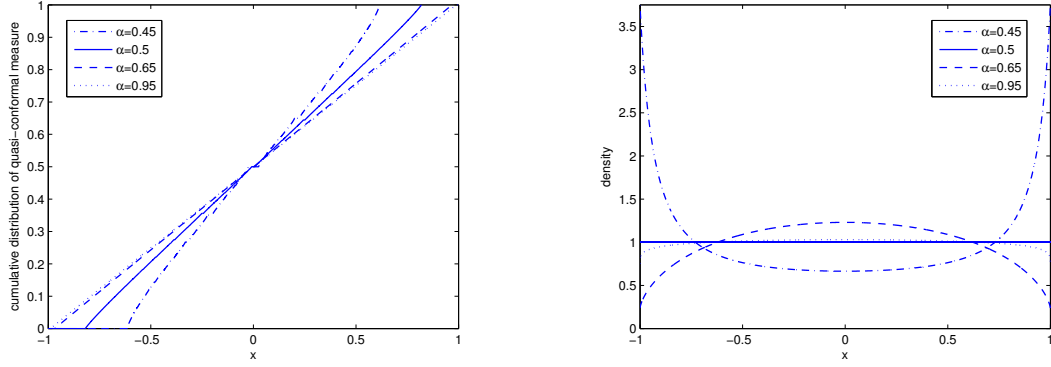


FIGURE 6. Open Ulam approximations for $T_{c,\alpha}$ ($k = 20000$). Left: cumulative distribution functions for $\mu_{2.01,\alpha}$ where $\alpha = 0.45, 0.5, 0.65, 0.95$. Right: accims for $T_{2.01,\alpha}$ (same α).

approximate accims agree (modulo scaling) between $\pm y_{c,\alpha}$, the only difference is that the different X_0 s lead to a different concentration of mass on preimages of the hole. The approximated escape rates are displayed in Figure 5, and concentration of accim on the hole (neighbourhoods of ± 1) is evident in Figure 6 (right). Note also that Figure 6 (left) depicts some approximate quasiconformal measures for $c = 2.01$ and $\alpha < 0.45$.

5. PROOFS

5.1. Auxiliary lemmas. Under the assumptions of Theorem 2.6, the quasi-conformal measure μ of (\hat{T}, H_0) satisfies some further properties that will be exploited in our approach. The measure μ can be used to define a useful cone of functions in BV . For each $a > 0$ let

$$\mathcal{C}_a = \{0 \leq f \in BV : \text{var}(f) \leq a\mu(f)\}.$$

Combining the result of Lemmas 4.2 and 4.3 from [21] with the argument in the proof of Lemma 3.7 (therein), the conditions on T imply the existence of a constant $a_1 > 0$ such that for any $a > a_1$ there is an $\epsilon_a > 0$ and $N \in \mathbb{N}$ such that

$$(7) \quad \mathcal{L}^N \mathcal{C}_a \subseteq \mathcal{C}_{a-\epsilon_a}.$$

The values of N , a_1 and ϵ_a are all computable in terms of the constants associated with T . We present a modified version of these arguments, based on the classical work of Rychlik [27], that specialize to the case $N = 1$, and allow us to improve some of the constants involved in the estimates of [21].

Lemma 5.1. *Let (\hat{T}, H_0) be a Lasota-Yorke map with an ϵ -Ulam-admissible hole. Then, there exists $K_\epsilon > 0$ such that for every $f \in BV$,*

$$\text{var}(\mathcal{L}f) \leq \alpha_\epsilon \text{var}(f) + K_\epsilon \mu(|f|).$$

Furthermore, there is a constant $a_1 > 0$ such that for any $a > a_1$ there is an $\epsilon_a > 0$ such that

$$(8) \quad \mathcal{L}\mathcal{C}_a \subseteq \mathcal{C}_{a-\epsilon_a}.$$

Proof. Let \mathcal{Z} be the monotonicity partition for \hat{T} . Define $\hat{g} : I \rightarrow \mathbb{R}$ by $\hat{g}(x) = |D\hat{T}(x)|^{-1}$ for every $x \in \left(I \setminus \bigcup_{Z \in \mathcal{Z}} \partial Z\right) \cup \{0, 1\}$, and $\hat{g}(x) = 0$ otherwise. We obtain the following

Lasota-Yorke inequality by adapting the approach of Rychlik [27, Lemmas 4-6]. Let $\mathcal{Z}_\epsilon \in \mathcal{G}_\epsilon$. Then,

$$\text{var}(\hat{\mathcal{L}}f) \leq \text{var}(f\hat{g}) \leq (2 + \epsilon)\|D\hat{T}^{-1}\|_\infty \text{var}(f) + \|D\hat{T}^{-1}\|_\infty(1 + \epsilon) \sum_{A \in \mathcal{Z}_\epsilon} \inf_A |f|.$$

We slightly modify \hat{g} to account for the jumps at the hole H_0 , and define $g : I \rightarrow \mathbb{R}$ by $g = 1_{X_0}\hat{g}$. Now, only elements of \mathcal{Z}_ϵ^* contribute to the variation of $\hat{\mathcal{L}}f$, and we get

$$\begin{aligned} \text{var}(\mathcal{L}f) &= \text{var}(\hat{\mathcal{L}}(1_{X_0}f)) \leq \text{var}(f(1_{X_0}\hat{g})) = \sum_{A \in \mathcal{Z}_\epsilon^*} \text{var}_A(f(1_{X_0}\hat{g})) \\ &\leq \sum_{A \in \mathcal{Z}_\epsilon^*} \text{var}_A(f)\|1_{X_0}\hat{g}\|_\infty + \|1_A f\|_\infty \text{var}_A(1_{X_0}\hat{g}) \\ &\leq \sum_{A \in \mathcal{Z}_\epsilon^*} \text{var}_A(f)\|DT^{-1}\|_\infty + \left(\inf_A |f| + \text{var}_A(f) \right) \text{var}_A(g). \end{aligned}$$

Thus, since for every $A \in \mathcal{Z}_\epsilon^*$, $\text{var}_A(g) \leq \|DT^{-1}\|_\infty(1 + \epsilon)$, one has that

$$(9) \quad \text{var}(\mathcal{L}f) \leq (2 + \epsilon)\|DT^{-1}\|_\infty \text{var}(f) + \sum_{A \in \mathcal{Z}_\epsilon^*} \|DT^{-1}\|_\infty(1 + \epsilon) \inf_A |f|.$$

Now we proceed as in the proof of [21, Lemma 2.5], and observe that there exists $\delta > 0$ such that if $A \in \mathcal{Z}_{\epsilon,g}$, then

$$(10) \quad \inf_A |f| \leq \delta^{-1} \mu(1_A |f|),$$

whereas if $A \in \mathcal{Z}_{\epsilon,b}$, we let $A' \in \mathcal{Z}_{\epsilon,g}$ be the nearest good partition element⁷, and get

$$\inf_A |f| \leq \inf_{A'} |f| + \text{var}_{I(A,A')}(f),$$

where $I(A, A')$ is an interval that contains A and has as an endpoint $x_{A'} \in A'$, fixed in advance, such that, after possibly redefining f at the discontinuity points of f , $|f(x_{A'})| = \inf_{A'} |f|$. Notice that either $I(A, A') \subseteq I_-(A')$ or $I(A, A') \subseteq I_+(A')$, where $I_+(A')$ is the union of $A'_+ := A' \cap \{x : x \geq x_{A'}\}$ with the contiguous elements of $\mathcal{Z}_{\epsilon,b}$ on the right of A' , and $I_-(A')$ is defined in a similar manner. Thus,

$$(11) \quad \sum_{A \in \mathcal{Z}_{\epsilon,b}} \inf_A |f| \leq \xi_\epsilon \text{var}(f) + 2\xi_\epsilon \sum_{A' \in \mathcal{Z}_{\epsilon,g}} \inf_{A'} |f|,$$

where the factor 2 appears due to the fact that a single good interval could have at most ξ_ϵ bad intervals on the left and ξ_ϵ bad intervals on the right. Combining equations (10) and (11), we get

$$\sum_{A \in \mathcal{Z}_\epsilon^*} \inf_A |f| \leq \xi_\epsilon \text{var}(f) + \delta^{-1}(1 + 2\xi_\epsilon) \sum_{A' \in \mathcal{Z}_{\epsilon,g}} \mu(1_{A'} |f|).$$

Plugging back into (9), we get

$$\text{var}(\mathcal{L}f) \leq \|DT^{-1}\|_\infty(2 + \epsilon + \xi_\epsilon) \text{var}(f) + \|DT^{-1}\|_\infty(1 + \epsilon)\delta^{-1}(1 + 2\xi_\epsilon)\mu(|f|).$$

We get the first part of the lemma by choosing $K_\epsilon = \|DT^{-1}\|_\infty(1 + \epsilon)\delta^{-1}(1 + 2\xi_\epsilon)$. For the second part, we recall that $\mu(\mathcal{L}f) = \rho\mu(f)$, so for every $f \in \mathcal{C}_a$, we have that

$$\frac{\text{var}(\mathcal{L}f)}{\mu(\mathcal{L}f)} \leq \frac{\alpha_\epsilon}{\rho} a + \frac{K_\epsilon}{\rho}.$$

⁷It is shown in [21, Lemma 2.4] that whenever (T, H_0) is an open system with an admissible hole, then $\mathcal{Z}_{\epsilon,g} \neq \emptyset$

Thus, $\mathcal{L}f \in \mathcal{C}_a$, provided $a > \frac{K_\epsilon}{\rho - \alpha_\epsilon} =: a_1$. \square

Moving toward a $BV, L^1(\text{Leb})$ Lasota-Yorke inequality, we have the following.

Lemma 5.2. *Let $\zeta > 0$ be given. Then there is a constant $B_\zeta < \infty$ such that*

$$\mu(f) \leq B_\zeta |f|_1 + \zeta \text{var}(f),$$

for $0 \leq f \in BV(I)$.

Proof. Let $\mathcal{Z}^{(n)}$ be the n -fold monotonicity partition for T_0 where n is such that $\mu(Z) < \frac{\zeta}{2}$ for all $Z \in \mathcal{Z}^{(n)}$. This choice is possible in view of [21, Lemma 3.10]. Choose k such that every $\frac{1}{k}$ subinterval intersects at most two such Z . Then, if K is a subinterval of length $1/k$, there are elements $Z_1, Z_2 \in \mathcal{Z}^{(n)}$ such that $K \subset Z_1 \cup Z_2$; hence $\mu(K) < \zeta = \zeta k m(K)$. Now let ξ be a partition of I into subintervals of length $1/k$ and put

$$F = \sum_{K \in \xi} \text{ess sup}_K f \mathbf{1}_K.$$

Then, $f \leq F$ and $F - f \leq \sum_{K \in \xi} V_K(f) \mathbf{1}_K$, where $V_K(f)$ denotes the variation of f inside the interval K . Thus,

$$|F - f|_1 \leq \sum_{K \in \xi} V_K(f) m(K) = V_I(f)/k.$$

We now estimate

$$\begin{aligned} \int f d\mu &\leq \int F d\mu = \sum_{K \in \xi} \text{ess sup}_K f \mu(K) \\ &\leq \sum_{K \in \xi} \text{ess sup}_K f \zeta \\ &= \zeta k |F|_1 \\ &= \zeta k |f|_1 + \zeta k |F - f|_1 \\ &\leq \zeta k |f|_1 + \zeta V_I(f). \end{aligned}$$

Putting $B_\zeta = \zeta k$ completes the proof. \square

A direct consequence of Lemmas 5.1 and 5.2 is the following.

Corollary 5.3. *Let $\alpha_\epsilon < \alpha < \rho$, where α_ϵ is defined in Equation (3). Then, there exists $K > 0$ such that*

$$(12) \quad \text{var}(\mathcal{L}f) \leq \alpha \text{var}(f) + K|f|_1.^8$$

Proof. Let $\zeta' = \frac{\alpha - \alpha_\epsilon}{K_\epsilon}$, where K_ϵ comes from Lemma 5.1. Let $B_{\zeta'}$ be given by Lemma 5.2. Then, Lemma 5.1 ensures

$$\begin{aligned} \text{var}(\mathcal{L}f) &\leq \alpha_\epsilon \text{var}(f) + K_\epsilon(B_{\zeta'}|f|_1 + \zeta' \text{var}(f)) \\ &= \alpha \text{var}(f) + K|f|_1. \end{aligned}$$

\square

Another useful result regarding the relation between the Ulam approximations and the accim and quasi-conformal measure is the following.

⁸For convenience, we have dropped the ϵ dependence on α and K . This should cause no confusion in the sequel, as ϵ is fixed throughout the section.

Lemma 5.4. *There exists $n > 0$ such that $(P_k^n)_{ij} > 0$ for all i, j satisfying $\mu(I_i) > 0$ and $\int_{I_j} h \, dm > 0$.*

Proof. Fix i, j satisfying the hypotheses. By Theorem 2.6, $\|(\mathcal{L}^n \chi_i)/\rho^n - \mu(I_i)h\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Choose n_{ij} large enough so that $\int_{I_j} \mathcal{L}^N \chi_i \, dm > 0$ for all $N \geq n_{ij}$. Because there are a finite number of I_i and I_j we can put $n = \max_{i,j} n_{ij}$ and obtain $\int_{I_j} \mathcal{L}^n \chi_i \, dm > 0$ for all i, j satisfying the hypotheses. Note that this implies $\int_{I_j} (\pi_k \mathcal{L})^n \chi_i \, dm > 0$ because the support of the integrand is possibly enlarged by taking Ulam projections. This now implies $(P_k^n)_{ij} > 0$. \square

5.2. Proof of the main result. The lemmas presented in §5.1 allow us to derive parts (I) and (II) of Theorem 3.2 via the perturbative approach from [19]. Indeed, Theorem 2.6 shows that $\rho > \alpha$ is the leading eigenvalue of \mathcal{L} , and that it is simple. Furthermore, \mathcal{L}_k is a small perturbation of \mathcal{L} for large k , in the sense that $\sup_{\|f\|_{BV}=1} |(\mathcal{L}_k - \mathcal{L})f|_1 \rightarrow 0$ as $k \rightarrow \infty$. Indeed,

$$\begin{aligned} \sup_{\|f\|_{BV}=1} |(\mathcal{L}_k - \mathcal{L})f|_1 &= \sup_{\|f\|_{BV}=1} |(\pi_k - Id)\mathcal{L}f|_1 \leq \sup_{\|f\|_{BV}=\|\mathcal{L}\|_{BV}} |(\pi_k - Id)f|_1 \\ &\leq \|\mathcal{L}\|_{BV} \max_{I_j \in \mathcal{P}_k} m(I_j), \end{aligned}$$

and the latter is proportional to τ_k , the diameter of the partition, which tends to 0 as $k \rightarrow \infty$.

Since π_k decreases variation, Corollary 5.3 implies the uniform inequality

$$(13) \quad \text{var}(\mathcal{L}_k f) \leq \alpha \text{var}(f) + K|f|_1, \quad \forall k \in \mathbb{N},$$

which is the last hypothesis to check to be in the position to apply the perturbative machinery of [19]. This result ensures that for sufficiently large k , \mathcal{L}_k has a simple eigenvalue ρ_k near ρ , and its corresponding eigenvector $h_k \in BV$ converges to h in $L^1(\text{Leb})$, giving the convergence statements in (I) and (II).

In order to show (III), we consider the operator $\bar{\mathcal{L}}_k := \mathcal{L}_k \circ \pi_k$. In view of Lemma 3.1, $\bar{\mathcal{L}}_k^* \mu_k = \rho_k \mu_k$, and $\bar{\mathcal{L}}_k h_k = \rho_k h_k$. As in the previous paragraph, one can check that $\bar{\mathcal{L}}_k$ is a small perturbation of \mathcal{L} . In fact,

$$\sup_{\|f\|_{BV}=1} |(\bar{\mathcal{L}}_k - \mathcal{L})f|_1 \leq 2 \max_{I_j \in \mathcal{P}_k} m(I_j) = 2\tau_k.$$

Also, the Lasota-Yorke inequality (12) holds with \mathcal{L} replaced by $\bar{\mathcal{L}}_k$. Thus, [19, Corollary 1] (see (iii) below) shows that for large k , ρ_k is the leading eigenvalue of $\bar{\mathcal{L}}_k$.

Let Π_k be the spectral projectors defined by

$$\Pi_k := \frac{1}{2\pi i} \oint_{\partial B_\delta(\rho)} (z - \bar{\mathcal{L}}_k)^{-1} dz,$$

where δ is small enough to exclude all spectrum of \mathcal{L} apart from the peripheral eigenvalue ρ . Also let

$$\Pi_0 := \frac{1}{2\pi i} \oint_{\partial B_\delta(\rho)} (z - \mathcal{L})^{-1} dz.$$

Then, [19, Corollary 1] provides $K_1, K_2 > 0$, and $\eta \in (0, 1)$ for which

- (i) $|(\Pi_k - \Pi_0)f|_1 \leq K_1 \tau_k^\eta \|f\|_{BV}$,
- (ii) $\|\Pi_k f\|_{BV} \leq K_2 |\Pi_k f|_1$,
- (iii) For large enough k , $\text{rank}(\Pi_k) = \text{rank}(\Pi_0)$.

Since ρ is simple and isolated, this setup implies that for large enough k , each Π_k is a bounded, rank-1 operator on BV :

$$\Pi_k = \mu_k(\cdot) h_k,$$

where each $h_k \in BV$, $\bar{\mathcal{L}}_k h_k = \rho_k h_k$ and $\rho_k \in B_\delta(\rho)$. Since $h_k = \Pi_k h_k$ we can choose $|h_k|_1 = 1$ so that $\|h_k\|_{BV} \in [1, K_2]$. Now, let $g \in BV$. Then, by the above,

$$\begin{aligned} |\mu_k(g) - \mu(g)| &= |(\mu_k(g) - \mu(g))h_k|_1 \leq |\mu_k(g)h_k - \mu(g)h|_1 + |\mu(g)(h_k - h)|_1 \\ &= |\Pi_k(g) - \Pi_0(g)|_1 + |\mu(g)| |h_k - h|_1 \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Since μ and μ_k are in fact measures, the above is enough to show that $\mu_k \rightarrow \mu$ in the weak-* topology.

In particular, there is a k_0 such that $\mu_k(h) > 0$ for all $k \geq k_0$. To show the last claim of (III), we will show that if $\mu_k(h) > 0$ then $\text{supp}(\mu) \subseteq \text{supp}(\mu_k)$. Let ψ_k be a leading right eigenvector of $\bar{\mathcal{L}}_k$ such that $P_k \psi_k = \rho_k \psi_k$ and $[\psi_k]_l = \frac{\mu(I_l)}{m(I_l)}$ ($l = 1, \dots, k$). Choose i such that $\mu(I_i) > 0$, j such that $[\psi_k]_j = \int_{I_j} h dm = \int_{I_j} h d\mu_k > 0$ and $n \geq n_{ij}$ as in Lemma 5.4. Then,

$$[\psi_k]_i = \rho^{-n} [P_k^n \psi_k]_i \geq \rho^{-n} [P_k^n]_{ij} [\psi_k]_j > 0.$$

This establishes that $\mu_k(I_i) > 0$ and hence that $\text{supp}(\mu) \subseteq \cup\{I_i : \mu(I_i) > 0\} \subset \text{supp}(\mu_k)$, as claimed.

For the quantitative statement of (I), note that for every $f \in BV$, $0 = (\mathcal{L} - \rho I)h = (\mathcal{L} - \rho I)\Pi_0 f$, so that

$$(\rho_k - \rho)h_k = (\bar{\mathcal{L}}_k - \mathcal{L})h_k + (\mathcal{L} - \rho)(\Pi_k - \Pi_0)h_k.$$

Hence,

$$\begin{aligned} |\rho_k - \rho| |h_k|_1 &\leq 2\tau_k \|h_k\|_{BV} + (|\mathcal{L}|_1 + |\rho|) K_1 \tau_k^\eta \|h_k\|_{BV} \\ &\leq 2(\tau_k + (1 + |\rho|) K_1 \tau_k^\eta) K_2 |h_k|_1, \end{aligned}$$

where K_1, K_2 and η are as above. This gives the error bound $|\rho_k - \rho| \leq O(\tau_k^\eta)$. \square

5.3. Proof of Lemma 2.8. Let \mathcal{L}_m be the transfer operator associated to T_m . That is, $\mathcal{L}_m(f) = \hat{\mathcal{L}}(1_{X_m} f)$. Then, $\mathcal{L}_m^n(f) = \hat{\mathcal{L}}^n(1_{X_{m+n-1}} f)$, and therefore,

$$(14) \quad \hat{\mathcal{L}}^m \circ \mathcal{L}_m^n = \mathcal{L}_0^{m+n}.$$

Hence, an interval is good for T_0 if and only if it is good for T_m for every m . In the rest of this proof we will say an interval is good if it is good for either (and therefore all) T_m .

Let $\mathcal{Z}_0 = \mathcal{Z} \vee \mathcal{H}_0$, where \mathcal{H}_0 is the partition of H_0 into intervals, and we recall that \mathcal{Z} is the monotonicity partition of \hat{T} . Let \mathcal{G}_ϵ be an ϵ -adequate partition for T_0 . Then, a partition $\mathcal{G}_{\epsilon,m}$ may be constructed by cutting each element of $\mathcal{G}_\epsilon \vee \mathcal{Z}_0^{(m)}$ in at most K pieces, where K is independent of m , in such a way that the variation requirement $\max_{Z \in \mathcal{G}_{\epsilon,m}} \text{var}_Z(\hat{g}1_{X_m}) \leq \|DT_m^{-1}\|_\infty (1 + \epsilon)$ is satisfied, and thus $\mathcal{G}_{\epsilon,m}$ is an ϵ -adequate partition for T_m . Indeed, $K = 2 + \lceil \|\hat{g}\|_\infty / \text{essinf}(\hat{g}) \rceil$ is a possible choice. The term 2 allows one to account for possible jumps at the boundary points of H_m , as there are at most two of them in each $Z \in \mathcal{G}_\epsilon \vee \mathcal{Z}_0^{(m)}$. The term $M = \lceil \|\hat{g}\|_\infty / \text{essinf}(\hat{g}) \rceil$ allows one to split each interval $Z \in \mathcal{G}_\epsilon \vee \mathcal{Z}_0^{(m)}$ into at most M subintervals Z_1, \dots, Z_M , in such a way that for every $1 \leq j \leq M$, $\text{var}_{\text{int}(Z_j)}(\hat{g}1_{X_m}) \leq (1 + \epsilon) \|\hat{g}1_{X_m}\|_\infty$. The chosen value of M is necessary to account for the possible discrepancy between $\|\hat{g}1_{X_0}\|_\infty$ and $\|\hat{g}1_{X_m}\|_\infty$. (Recall also that \hat{g} is continuous on each $\text{int}(Z_j)$.)

Now, let $b = \#\mathcal{Z}_0$. Then, each bad interval of \mathcal{G}_ϵ gives rise to at most Kb^m (necessarily bad) intervals in $\mathcal{G}_{\epsilon,m}$. When a good interval of \mathcal{G}_ϵ is split, it also gives rise to at most Kb^m

intervals in $\mathcal{G}_{\epsilon,m}$. In this case some of the intervals may be bad, but it is guaranteed that at least one of them remains good, as being good is equivalent to having non-zero μ measure. Thus, the number of contiguous bad intervals in $\mathcal{G}_{\epsilon,m}$ is at most $Kb^m(B+2)$, where B is the number of contiguous bad intervals in \mathcal{G}_ϵ . Therefore, $\tilde{\xi}_\epsilon(T_m) = \exp\left(\limsup_{n \rightarrow \infty} \frac{1}{n} \log(1 + \xi_{\epsilon,n}(T_m))\right) \leq \tilde{\xi}_\epsilon(T_0)$.

Clearly, $\tilde{\Theta}(T_m) \leq \tilde{\Theta}(T_0)$. Finally, we will show that $\rho(T_0) \leq \rho(T_m)$. Recall that ρ_j is the leading eigenvalue of \mathcal{L}_j . Let $f \in BV$ be nonzero and such that $\mathcal{L}_0 f = \rho_0 f$. We claim that $\mathcal{L}_m(1_{X_{m-1}} f) = \rho_0 1_{X_{m-1}} f$, which yields the inequality, because necessarily $1_{X_{m-1}} f \neq 0$. Indeed,

$$\rho_0 1_{X_{m-1}} f = 1_{X_{m-1}} \mathcal{L}_0 f = 1_{X_{m-1}} \mathcal{L}_m f = \mathcal{L}_m f = \mathcal{L}_m(1_{X_{m-1}} f),$$

where the second equality follows from the fact that $\mathcal{L}_0(1_{H_m} f)$ is supported on $T(H_m) = H_{m-1}$. The third one, from the fact that $\mathcal{L}_m f$ is supported on $T(X_m) \subseteq X_{m-1}$. The last one, because $\mathcal{L}_m(1_{H_{m-1}} f) = 0$.

The first statement of the lemma follows. The relations between escape rates, accims and quasi-conformal measures follow from comparing via Equation (14) the statements of part (4) of Theorem 2.6 applied to T_0 and T_m . \square

5.4. Proof of Lemma 3.3.

Assume H_0 is an ϵ -admissible hole for \hat{T} . Then, $T^n := (\hat{T}^n, H_{n-1})$ is an open Lasota-Yorke map. Fix $\tilde{\Theta} < \eta < \rho$ so that for all n sufficiently large,

$$\exp\left(\frac{1}{n} \log \|(DT^n)^{-1}\|_\infty\right) \exp\left(\frac{1}{n} \log(1 + \xi_{\epsilon,n})\right) < \eta.$$

Then, $\|(DT^n)^{-1}\|_\infty \xi_{\epsilon,n} < \eta^n$. By possibly making n larger, we can assume that $(2 + \epsilon)\|(DT^n)^{-1}\|_\infty < \eta^n$, and that $2\eta^n < \rho^n$. Then, $\|(DT^n)^{-1}\|_\infty(2 + \epsilon + \xi_{\epsilon,n}) < \rho^n$.

We remark that $\xi_\epsilon(T^n) = \xi_{\epsilon,n}(T)$. Thus $\alpha_\epsilon(T^n) = \|(DT^n)^{-1}\|_\infty(2 + \epsilon + \xi_{\epsilon,n})$. Furthermore, in view of Theorem 2.6, $\rho(T^n) = \lim_{m \rightarrow \infty} \inf_{x \in D_{mn}} \frac{\mathcal{L}^{n(m+1)} 1(x)}{\mathcal{L}^{nm} 1(x)} = \mu(\mathcal{L}^n 1) = \rho^n$. \square

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